REVERSE YIELDING OF A FULLY AUTOFRETTAGED TUBE OF LARGE WALL RATIO

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REVERSE YIELDING OF A FULLY AUTOFRETTAGED TUBE OF LARGE WALL RATIO

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ABSTRACT: The equations are developed for the case of a reverse yielded thick-walled cylinder. It is assumed that a cylinder is subjected to an internal pressure which causes plastic flow throughout the wall; the size of the cylinder is such that the residual stresses developed during pressure release cause the cylinder to reyield in compression. The stress equations for the subsequent reapplication of pressure to the reyielded cylinder are also developed.

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This report is the result of a need to provide high-strength chambers for use in hypervelocity launchers. The calculations presented provide understanding about the present limitation, and reverse yielding of autofrettaged cylinders.

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Captain, USN
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By direction

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ILLUSTRATIONS

REFERENCES

(1) Nadai, A., Plasticity, McGraw-Hill Book Company, New York

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(3) Dawson, V. C. D., "Elastic and Plastic Stress Equations for Hollow Cylinders," NavOrd Report 6786

LIST OF SYMBOLS

a Inner radius of cylinder

b Outer radius of cylinder

c Radius of interface between thrice yielded region (first in tension, then in compression, finally in tension) and the twice yielded region

d Radius of interface between once yielded region in tension and twice yielded region (once in tension, then in compression)

D Diameter

m Diameter ratio inside region where tube is elastic (i.e., m is greater than n)

n Diameter ratio to which plastic flow has occurred

p Internal pressure applied to cylinder after reverse yielding has occurred

P Internal pressure applied to cylinder before reverse yielding has occurred

r Radius

w Diameter ratio inside region where plastic flow has occurred (i.e., w is less than n)

Yo Yield strength

Yo_c Yield strength in compression

Yo_t Yield strength in tension

σ Stress

ω Wall ratio (b/a)

<u>Superscript</u>

* Residual (stress)

Subscripts

t	Tangential (stress)
r	Radial (stress)
z	Longitudinal (stress)
max	Maximum (stress)
i	Interface

INTRODUCTION

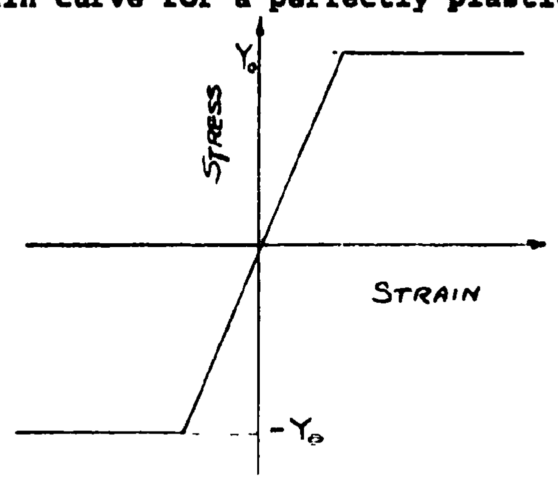
The pressure capability of a closed-end cylindrical pressure vessel is limited for elastic operation. Based upon the Distortion Energy Theory, the pressure at which yielding begins at the bore is given by

$$p = \frac{Y_0 (\omega^2 - 1)}{\sqrt{3}\ \omega^2} \tag{1}$$

Thus, even for very large wall ratios the maximum pressure a cylinder will hold elastically is given by $P = Y_0/\sqrt{3}$.

One of the methods of increasing the elastic pressure capability of a cylinder is the use of autofrettage. This process consists of inducing plastic flow in the cylinder during manufacture by pressurizing it with a pressure (the "autofrettage pressure") greater than that given by equation (1). The plastic flow of the metal begins at the bore and progresses through the wall as the pressure is increased. This non-uniform flow is such that when pressure is released, the wall is left with a residual stress distribution such that the bore has a compressive tangential stress. The cylinder is then said to be autofrettaged. Subsequent pressure application can be made up to the autofrettage pressure with the cylinder reacting elastically.

The equations for the autofrettage process (based on a perfectly plastic material) have been derived by numerous investigators (see for example, refs. (1), (2) and (3)). The stress-strain curve for a perfectly plastic material is sketched below.



According to reference (3) the pressure required to deform a cylinder of wall diameter ratio ω, plastically, to a diameter ratio n is

$$p = \frac{2Y_0}{\sqrt{3}}\left(\frac{\omega^2 - n^2}{2\omega^2} + \ln n\right) \tag{2}$$

Upon release of the pressure given by equation (2) the residual stress distribution is given by

$$\sigma_t^* = -p + \frac{2Y_0}{\sqrt{3}}\ln w + \frac{2Y_0}{\sqrt{3}} - \frac{p}{\omega^2-1}\left(1 + \frac{\omega^2}{w^2}\right) \qquad w \leq n \tag{3}$$

$$\sigma_r^* = -p + \frac{2Y_0}{\sqrt{3}}\ln w - \frac{p}{\omega^2-1}\left(1 - \frac{\omega^2}{w^2}\right) \qquad w \leq n \tag{4}$$

at any position diameter ratio w in the part of the tube that was plastically deformed. In the part that was elastic

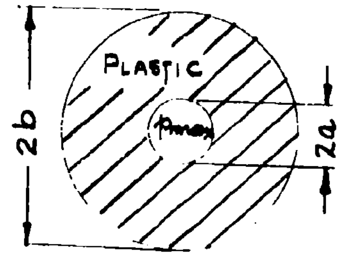
$$\sigma_t^* = \frac{Y_0}{\sqrt{3}}\left(\frac{n^2}{m^2} + \frac{n^2}{\omega^2}\right) - \frac{p}{\omega^2-1}\left(1 + \frac{\omega^2}{m^2}\right) \qquad \omega \geq m \geq n \tag{5}$$

$$\sigma_r^* = -\frac{Y_0}{\sqrt{3}}\left(\frac{n^2}{m^2} - \frac{n^2}{\omega^2}\right) - \frac{p}{\omega^2-1}\left(1 - \frac{\omega^2}{m^2}\right) \qquad \omega \geq m \geq n \tag{6}$$

where m is the position diameter ratio. It is assumed that the residual stresses at the bore are not large enough to cause the bore, which had previously been yielded in tension, to yield in compression, that is, to "reyield" or "reverse yield."

As the pressure is increased during autofrettage a point is reached where the tube is entirely plastic, i.e., $n = \omega$. This represents the maximum pressure which can be applied without rupturing the cylinder for a perfectly plastic material and is, according to equation (2),

$$p_{max} = \frac{2Y_0}{\sqrt{3}} \ln \omega \qquad (7)$$



The residual stresses in this case are given by inserting the value of P_{MAX} for p in equations (3) and (4).

From equation (7) it is apparent that as ω is increased the pressure required to cause plastic flow throughout the wall increases. This, in turn, produces larger and larger residual compressive stresses at the bore upon pressure release. For the fully plastic case, then, there is some particular wall ratio at which the residual stresses will be large enough to just cause yielding at the bore in compression upon release of the autofrettage pressure.

To determine the wall ratio at which the residual stresses at the bore of a fully autofrettaged cylinder are large enough to just cause it to yield in compression the yield criterion used in reference (3),

$$\sigma_t - \sigma_r = \frac{2Y_0}{\sqrt{3}} \quad * \qquad (8)$$

will be employed. Thus, the condition of yield, equation (8), becomes, when applied to the residual stresses at the bore,

$$\sigma_t^* - \sigma_r^* = -\frac{2Y_0}{\sqrt{3}} \qquad (9)$$

Here it has been assumed that the yield in compression is equal to the negative of the yield in tension.

Substituting the values of the residual stresses from equation (5) and equation (6) into equation (9) with $\bar{\omega}$ set equal to 1 (i.e., at the bore), one obtains

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*It was assumed that $\sigma_z = \frac{1}{2}(\sigma_r + \sigma_t)$ so that the yield condition becomes $\sigma_t - \sigma_r = \frac{2Y_0}{\sqrt{3}}$. See reference (2) for a discussion of the validity of this assumption.

$$\frac{2Y_0}{\sqrt{3}} - \frac{2p\omega^2}{\omega^2-1} = -\frac{2Y_0}{\sqrt{3}}$$

With $p_{max} = \frac{2Y_0}{\sqrt{3}} \ln \omega$ for the case of the fully autofrettaged cylinders, the equation above becomes

$$-\frac{2Y_0}{\sqrt{3}} = \frac{2Y_0}{\sqrt{3}} - \frac{4Y_0}{\sqrt{3}}\left(\frac{\omega^2}{\omega^2-1}\right) \ln \omega$$

or

$$\frac{\omega^2-1}{\omega^2} = \ln \omega \qquad (10)$$

Solving equation (10) gives $\omega = 2.22$. Thus, a cylinder having a wall ratio of 2.22, if autofrettaged to the fully plastic state, develops residual stresses of such magnitude that the bore is on the verge of reyielding (reverse yielding) in compression upon pressure release. If $\omega < 2.22$ the residual stresses developed are less than those required for reyielding for the fully plastic case and if $\omega > 2.22$ these stresses will cause reyielding for the fully plastic case.

It is further found that as the wall ratio increases above 2.22 the value of n to just leave the bore at the compressive yield limit decreases (ref. (3)). This means that, if the reyielding condition is the limiting design condition, there is a limit to the autofrettage pressure. This limit is calculated to be just twice the pressure to cause initial yielding at the bore. Hence, according to equation (1) the limiting autofrettage pressure for incipient reyielding in the case of $\omega > 2.2$ is

$$\frac{p}{Y_0} = \frac{2(\omega^2-1)}{\sqrt{3}\,\omega^2} \qquad (11)$$

For any cylinder there are, therefore, three limiting curves as shown in figure 1. These are the following:

$$\frac{p}{Y_0} = \frac{\omega^2-1}{\sqrt{3}\,\omega^2} \qquad (1)$$

the pressure at which yielding initially occurs;

$$\frac{p_{max}}{Y_o} = \frac{2}{\sqrt{3}} \ln \omega \tag{7}$$

the pressure necessary to make the cylinder fully plastic, which for $\omega \leq 2.22$, leaves the residual stresses low enough to prevent reyielding;

$$\frac{p}{Y_o} = \frac{2}{\sqrt{3}}\left(\frac{\omega^2-1}{\omega^2}\right) \tag{11}$$

the pressure limit for large wall ratios to just leave the bore at the yield point in compression after pressure release.

It is apparent that if a cylinder could be operated at pressures given by equation (7), a sizeable increase in pressure capability over that given in equation (11) would be possible. However, as noted before, in this circumstance, there would occur reyielding of the bore in compression when the pressure is released.

It is the purpose of this study to investigate reverse yielding in thick-walled cylinders that have been pressurized to the fully plastic state during autofrettage.

DERIVATION OF REYIELDING EQUATIONS ($\omega > 2.22$)*

The assumptions made are the following:

1. The material is assumed perfectly plastic
2. $\sigma_z = 1/2\,(\sigma_t + \sigma_r)$
3. The yield criterion is given by the Distortion Energy Theory

Assumptions (2) and (3) result in the following yield criterion:

$$\sigma_t - \sigma_r = \pm \frac{2Y_o}{\sqrt{3}} \tag{12}$$

Let us consider the case of the fully autofrettaged cylinder of $\omega > 2.2$ subjected to the pressure P_{MAX}, where

* See Appendix A for an alternate derivation of the reyielding equations.

$$\frac{p_{max}}{Y_o} = \frac{2}{\sqrt{3}} \ln \omega \tag{7}$$

At every point in the plastically deformed cylinder

$$\sigma_t - \sigma_r = \frac{2 Y_o}{\sqrt{3}} \tag{13}$$

The equation of equilibrium is

$$\sigma_t - \sigma_r = r \frac{d\sigma_r}{dr} \tag{14}$$

The radial stress at the bore is equal to $-P_{MAX}$, that is

$$\sigma_{r\,r=a} = -p_{max} = -\frac{2 Y_o}{\sqrt{3}} \ln \omega \tag{15}$$

Equations (13), (14), and (15) may be combined to obtain the plastic stresses due to the internal pressure P_{MAX} that exist in the cylinder (as was done in reference (3)). These stresses are

$$\sigma_r = -\frac{2 Y_o}{\sqrt{3}} \ln \frac{b}{r} \tag{16}$$

$$\sigma_t = -\frac{2 Y_o}{\sqrt{3}} \left(\ln \frac{b}{r} - 1 \right) \tag{17}$$

As the internal pressure is released, the cylinder deforms elastically until the bore reaches the yield point in compression. Thereafter, as the pressure is further reduced, plastic flow progresses outward from the bore. When the internal pressure reaches zero, the cylinder will consist of two zones, an inner core which has reyielded in compression (reverse yielded) and an outer elastic jacket that has been previously yielded in tension during autofrettage.

The equations describing the reyielded inner core when the pressure has been released are the yield criterion equation in compression, and the equilibrium equation (14), viz.,

$$\sigma_t - \sigma_r = -\frac{2Y_0}{\sqrt{3}} \tag{18}$$

$$\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \tag{14}$$

These equations with the boundary condition that the residual radial stress at the bore is zero lead to the following equations for the inner reyielded core stresses after pressure release:

$$\sigma_r^* = -\frac{2Y_0}{\sqrt{3}} \ln\frac{r}{a} \qquad a \leq r \leq d \tag{19}$$

$$\sigma_t^* = -\frac{2Y_0}{\sqrt{3}}\left(\ln\frac{r}{a} + 1\right) \qquad a \leq r \leq d \tag{20}$$

where "d" denotes the radius at the interface. These are the residual stresses after pressure release in the reyielded inner core.

The stresses in the outer jacket before pressure release are expressed by equation (16) and equation (17). Since during pressure release the outer jacket is only deformed elastically, the stresses may be obtained by superposition of elastic stresses. Thus,

$$\sigma_t^* \text{ after pressure release} = \sigma \text{ before pressure release} + \sigma \text{ due to change in effective pressure at the interface} \tag{21}$$

The change in effective pressure ΔP at the interface is given by:

ΔP = effective pressure at the interface after pressure has been released

minus

effective pressure at the interface before the pressure has been released

Since the effective pressure at the interface is equal to the negative of the radial stress at the interface, ΔP becomes

$$\Delta P = -\sigma_r \begin{array}{l}\text{after pressure} \\ \text{release at } r = d\end{array} + \sigma_r \begin{array}{l}\text{before pressure} \\ \text{release at } r = d\end{array}$$

From equations (16) and (17) the change in effective pressure becomes:

$$\Delta p = \frac{2Y_0}{\sqrt{3}} \ln \frac{d}{a} - \frac{2Y_0}{\sqrt{3}} \ln \frac{b}{d}$$

The elastic stresses due to a pressure ΔP at radius d are given by the equations (see reference (3) for example):

$$\sigma_r = \frac{-\Delta p}{\left(\frac{b}{d}\right)^2 - 1} \left(\left(\frac{b}{r}\right)^2 - 1\right) \tag{23}$$

$$\sigma_t = \frac{\Delta p}{\left(\frac{b}{d}\right)^2 - 1} \left(\left(\frac{b}{r}\right)^2 + 1\right) \tag{24}$$

Inserting equations (23), (24), (16), and (17) into (21) the expressions for the stresses in the once yielded jacket become

$$\sigma_r^* = -\frac{2Y_0}{\sqrt{3}} \left\{ \ln \frac{b}{r} + \left(\frac{(b/r)^2 - 1}{(b/d)^2 - 1} \right) \left(\ln \frac{d}{a} - \ln \frac{b}{d} \right) \right\} \tag{25}$$

$$b \geq r \geq d$$

$$\sigma_t^* = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - 1 - \left(\frac{(b/r)^2+1}{(b/d)^2-1}\right)\left(\ln\frac{d}{a} - \ln\frac{b}{d}\right)\right\} \qquad (26)$$

$$b \geqslant r \geqslant d$$

Equations (25) and (26) are thus the residual stresses in the outer once yielded jacket after the pressure has been released.

Since the tangential stress at the interface $r = d$ must be equal in each zone, one obtains by equating equation (26) to equation (20):

$$\ln\frac{d}{a} = \left(\frac{d}{b}\right)^2 - 1 + \ln\frac{b}{d} \qquad (27)$$

The extent of the reverse yielding can be calculated from this equation by solving for the inner core radius d.

The residual stresses may be rewritten by inserting equation (27) into equations (25) and (26) to give:

$$\sigma_r^* = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - \left(\frac{d}{r}\right)^2 - \left(\frac{d}{b}\right)^2\right\} \qquad (28)$$

$$b \geqslant r \geqslant d$$

$$\sigma_t^* = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - 1 + \left(\frac{d}{r}\right)^2 - \left(\frac{d}{b}\right)^2\right\} \qquad (29)$$

$$b \geqslant r \geqslant d$$

Equations (19), (20), (28), and (29) are thus the residual stresses developed in a fully autofrettaged thick-walled cylinder (i.e., $\omega > 2.2$) after pressure release. The extent of the reversed yielded plastic core of radius d is obtained from equations (27).

Figure 2 shows the plastic and elastic zone dimensions of a cylinder of wall ratio ω equal to 5. The value of d/a is calculated from equation (27) to be 1.41. It is seen that the plastic core is relatively small. Figure 3 is a plot of d/a for various wall ratios

Figure 4 shows the residual stress distribution (σ_r^*, σ_t^*) in a reyielded cylinder of $\omega = 5$. Included in the plot, as dotted lines, are the residual σ_r^* and σ_t^* that would exist if the cylinder had no limiting compressive yield strength. It can be seen that these residual stresses are only slightly modified in the elastic zone.

The results indicate that a cylinder with wall ratio greater than 2.22, if autofrettaged to the fully plastic condition, will have a reyielded core after pressure release. This plastic core has relatively small dimensions compared to the original dimensions of the cylinder.

PRESSURE APPLICATION TO THE REVERSE YIELDED CYLINDER

If pressure, p, is reapplied to the reverse yielded cylinder, then the core will initially deform elastically. However, if the pressure becomes sufficiently high, the tensile stresses in the core will cause it to begin to yield in tension at the bore. Further pressurization will cause the region of plastic deformation to extend radially from the bore to, say, a radius "c". For this plastic region the yield criterion

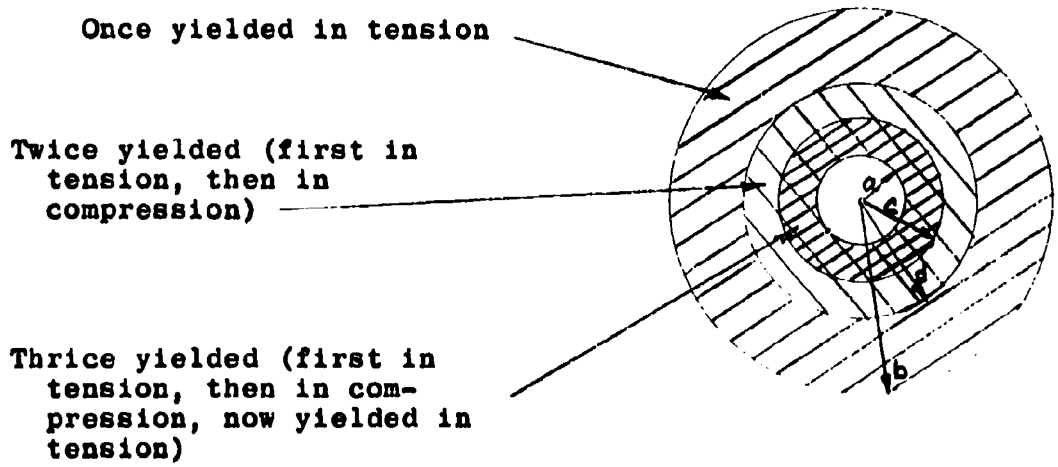
$$\sigma_t - \sigma_r = \frac{2Y_0}{\sqrt{3}} ,$$

the equilibrium equation (14), and the boundary condition that the radial stress at the bore is equal to minus the applied pressure, result in the following:

$$\sigma_r = -p + \frac{2Y_0}{\sqrt{3}} \ln \frac{r}{a} \qquad a \leq r \leq c \qquad (30)$$

$$\sigma_t = -p + \frac{2Y_0}{\sqrt{3}} \left(\ln \frac{r}{a} + 1 \right) \qquad a \leq r \leq c \qquad (31)$$

These are the stresses in the thrice yielded core of the cylinder. The cylinder at this time appears as sketched below.



Since the deformations in the cylinder other than in the thrice yielded core are elastic, the stresses may be obtained in these elastically deforming regions by the use of superposition. Thus, for the regions of radii greater than r = c,

$\sigma = \sigma$ before pressure application $+ \ \sigma$ due to change in effective pressure at the interface

The stresses before pressure application are the residual stresses; the change in the effective pressure at the interface is equal to the negative of the radial stress change at the interface. Thus, the above expression becomes

$\sigma = \sigma^* + \sigma$ due to effective pressure of value $\sigma_r^* - \sigma_r$ at interface

Hence, using equations (23) and (24) and denoting the interface radius by r_i

$$\sigma_r = \sigma_r^* - \frac{\sigma^*_{r_{r=r_i}} - \sigma_{r_{r=r_i}}}{\left(\frac{b}{r_i}\right)^2 - 1}\left(\left(\frac{b}{r}\right)^2 - 1\right) \tag{32}$$

$$\sigma_t = \sigma_t^* + \frac{\sigma_{r_{r=r_i}}^* - \sigma_{r_{r=r_i}}}{\left(\frac{b}{r_i}\right)^2 - 1}\left(\left(\frac{b}{r}\right)^2 + 1\right) \tag{33}$$

For the region

$$d \geqslant r \geqslant c$$

equations (32) and (33) become with $r_i = c$

$$\sigma_r = -\frac{2Y_o}{\sqrt{3}}\ln\frac{r}{a} - \left(\frac{p - \frac{4Y_o}{\sqrt{3}}\ln\frac{c}{a}}{\left(\frac{b}{c}\right)^2 - 1}\right)\left(\left(\frac{b}{r}\right)^2 - 1\right) \tag{34}$$

$$\sigma_t = -\frac{2Y_o}{\sqrt{3}}\left(\ln\frac{r}{a} + 1\right) + \left(\frac{p - \frac{4Y_o}{\sqrt{3}}\ln\frac{c}{a}}{\left(\frac{b}{c}\right)^2 - 1}\right)\left(\left(\frac{b}{r}\right)^2 + 1\right) \tag{35}$$

Equating the tangential stresses equation (35) and equation (31) at the interface r = c, one obtains an expression for the applied pressure, p, in terms of the radius, c, viz.,

$$\frac{\sqrt{3}}{2Y_o}p = 1 + 2\ln\frac{c}{a} - \left(\frac{c}{b}\right)^2 \tag{36}$$

If a = c is substituted into equation (36), there results

$$\frac{\sqrt{3}\,p}{2Y_o} = \frac{\omega^2 - 1}{\omega^2} \qquad a = c$$

which is identical to equation (11). Thus, the reverse yielded cylinder begins to yield a third time (at the bore surface) when the reapplied pressure is that given by equation (11); i.e., the cylinder, upon reapplication of pressure, withstands elastically the same pressure that it would have withstood if it had been autofrettaged in such a way as to leave the residual stresses at the bore at the compressive yield strength.

The stress equations (34) and (35) may be transformed by use of equation (36) to yield

$$\sigma_r = -\frac{2Y_0}{\sqrt{3}}\left(\ln\frac{r}{a} + \left(\frac{c}{r}\right)^2 - \left(\frac{c}{b}\right)^2\right) \qquad d \geq r \geq c \tag{37}$$

$$\sigma_t = -\frac{2Y_0}{\sqrt{3}}\left(\ln\frac{r}{a} + 1 - \left(\frac{c}{r}\right)^2 - \left(\frac{c}{b}\right)^2\right) \qquad d \geq r \geq c \tag{38}$$

For the region

$$b \geq r \geq d$$

equations (32) and (33) become with $r_1 = d$

$$\sigma_r = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - \left(\frac{d}{r}\right)^2 + \left(\frac{d}{b}\right)^2\right\} - \frac{2Y_0}{\sqrt{3}}\,\frac{\left(\frac{c}{b}\right)^2\left(\left(\frac{b}{d}\right)^2-1\right)}{\left(\frac{b}{d}\right)^2-1}\left(\left(\frac{b}{r}\right)^2-1\right)$$

$$\sigma_t = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - 1 + \left(\frac{d}{r}\right)^2 + \left(\frac{d}{b}\right)^2\right\} + \frac{2Y_0}{\sqrt{3}}\,\frac{\left(\frac{c}{b}\right)^2\left(\left(\frac{b}{d}\right)^2-1\right)\left(\left(\frac{b}{r}\right)^2+1\right)}{\left(\frac{b}{d}\right)^2-1}$$

These equations simplify to

$$\sigma_r = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - \left(\frac{d}{r}\right)^2 + \left(\frac{d}{b}\right)^2 + \left(\frac{c}{r}\right)^2 - \left(\frac{c}{b}\right)^2\right\} \qquad b \geq r \geq d \tag{39}$$

$$\sigma_t = -\frac{2Y_0}{\sqrt{3}}\left\{\ln\frac{b}{r} - 1 + \left(\frac{d}{r}\right)^2 + \left(\frac{d}{b}\right)^2 - \left(\frac{c}{r}\right)^2 - \left(\frac{c}{b}\right)^2\right\} \qquad b \geq r \geq d \tag{40}$$

From equations (39) and (40)

$$\sigma_t - \sigma_r = -\frac{2Y_0}{\sqrt{3}}\left(2\left(\frac{d}{r}\right)^2 - 2\left(\frac{c}{r}\right)^2 - 1\right) \qquad b \geq r \geq d$$

which for c = d becomes

$$\sigma_t - \sigma_r = \frac{2Y_0}{\sqrt{3}} \tag{41}$$

Equation (41) is independent of r and states that the outermost region which was elastic in the reyielded cylinder becomes plastic instantaneously when r reaches d during reapplication of the pressure. Also, from equations (36) and (27) with r = d

$$\frac{\sqrt{3}\,p}{2Y_0} = \ln\omega$$

which is simply equation (7), i.e., the pressure required to cause plastic flow throughout the entire wall. Thus, it is seen that upon reapplication of pressure, p, the inner bore begins to yield in tension for a third time when the pressure reaches the value

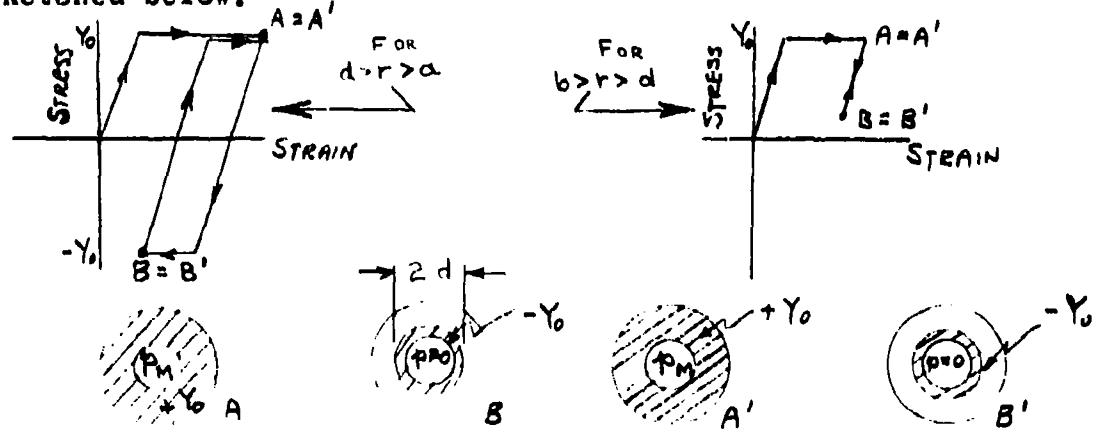
$$\frac{\sqrt{3}\,p}{2Y_0} = \frac{\omega^2-1}{\omega^2}$$

and the yielding progresses to larger radii as the pressure is increased. When the pressure reaches the value

$$\frac{\sqrt{3}\,p}{2Y_0} = \ln \omega$$

the yielding reaches the radius, d, at which time suddenly the entire wall becomes plastic.

The stress-strain history of elements in the tube wall is sketched below.



A plot of equation (36), giving the pressure required to extend the plastic zone when pressure is applied to a reyielded cylinder of wall ratio 5, is shown in figure 5.

CONCLUSIONS

Within the assumptions made, the equations for reyielding of a cylinder of large wall ratio ($\omega > 2.22$), autofrettaged to the fully plastic condition, have been derived. These equations indicate that the reyielded plastic zone has relatively small dimensions while the residual stresses in the outer elastic part of the tube are slightly altered from what they would have been had the inner core had no limiting compressive yield strength.

It has also been shown that subsequent application of pressure to the reyielded cylinder causes the bore to start yielding at the pressure that is the limit pressure for the autofrettaging of thick-walled cylinders. However, as pressure is built up the plastic zone grows but at a considorably slower rate than it did during the original autofrettage process. When the elastic-plastic interface reaches the outside radius of the original reyielded core, the entire cylinder becomes plastic.

It thus appears that the repetitively applied internal pressure capability of cylinders may be the fully autofrettaged pressure

$$p_{MAX} = \frac{2Y_0}{\sqrt{3}} \ln \omega$$

even for thick-walled cylinders where reverse yielding occurs (i.e., where $\omega > 2.2$). This conclusion requires experimental confirmation.

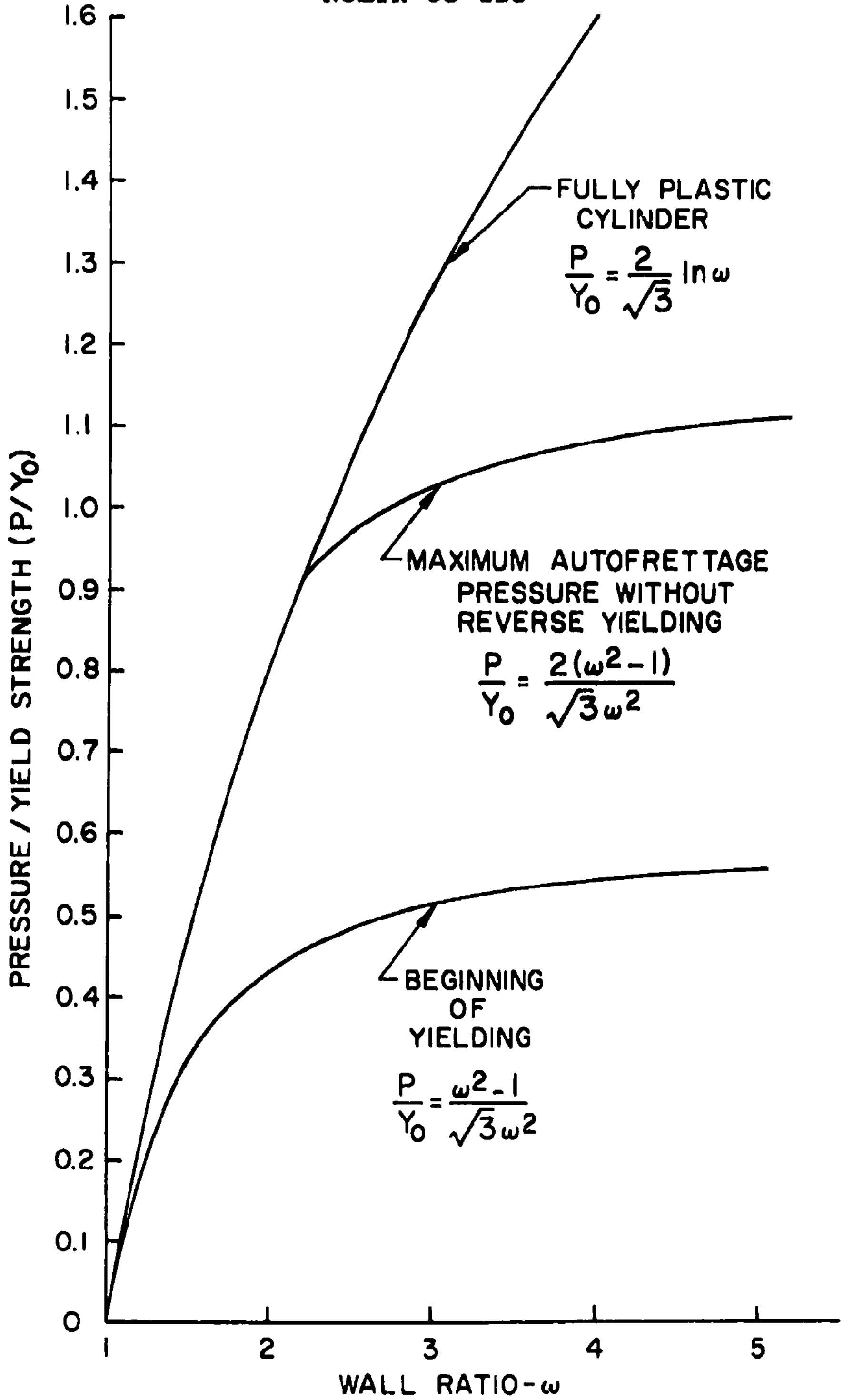


FIG. 1
LIMITING CURVES FOR INTERNAL PRESSURE APPLICATION TO A CYLINDER

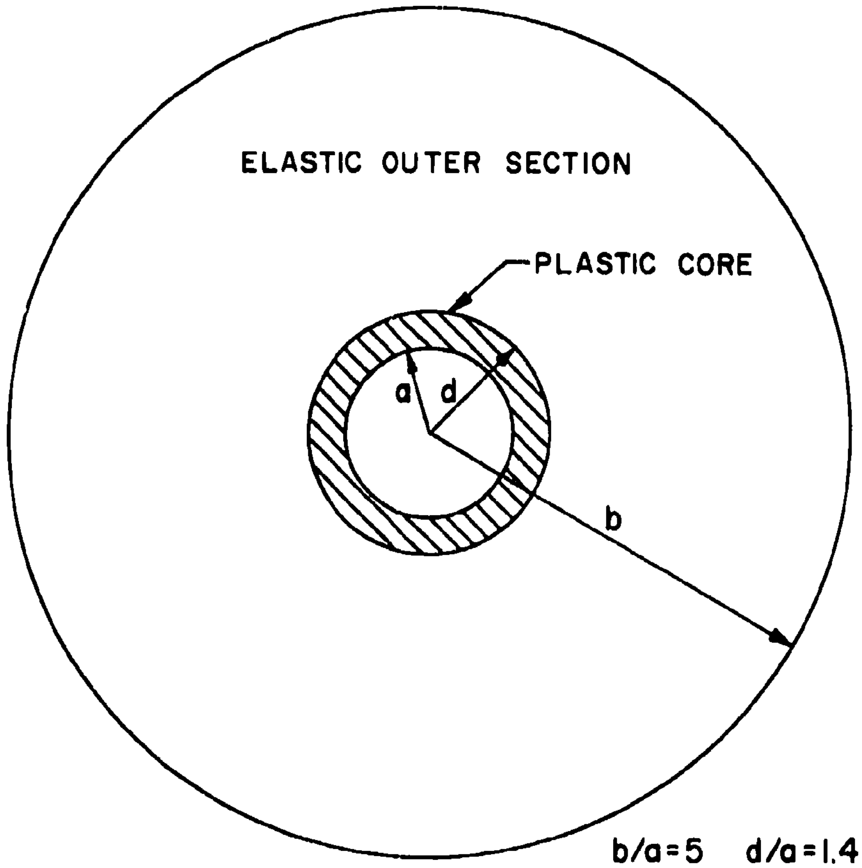


FIG. 2
SCALE DRAWING OF REYIELDED CYLINDER WITH WALL RATIO OF 5.

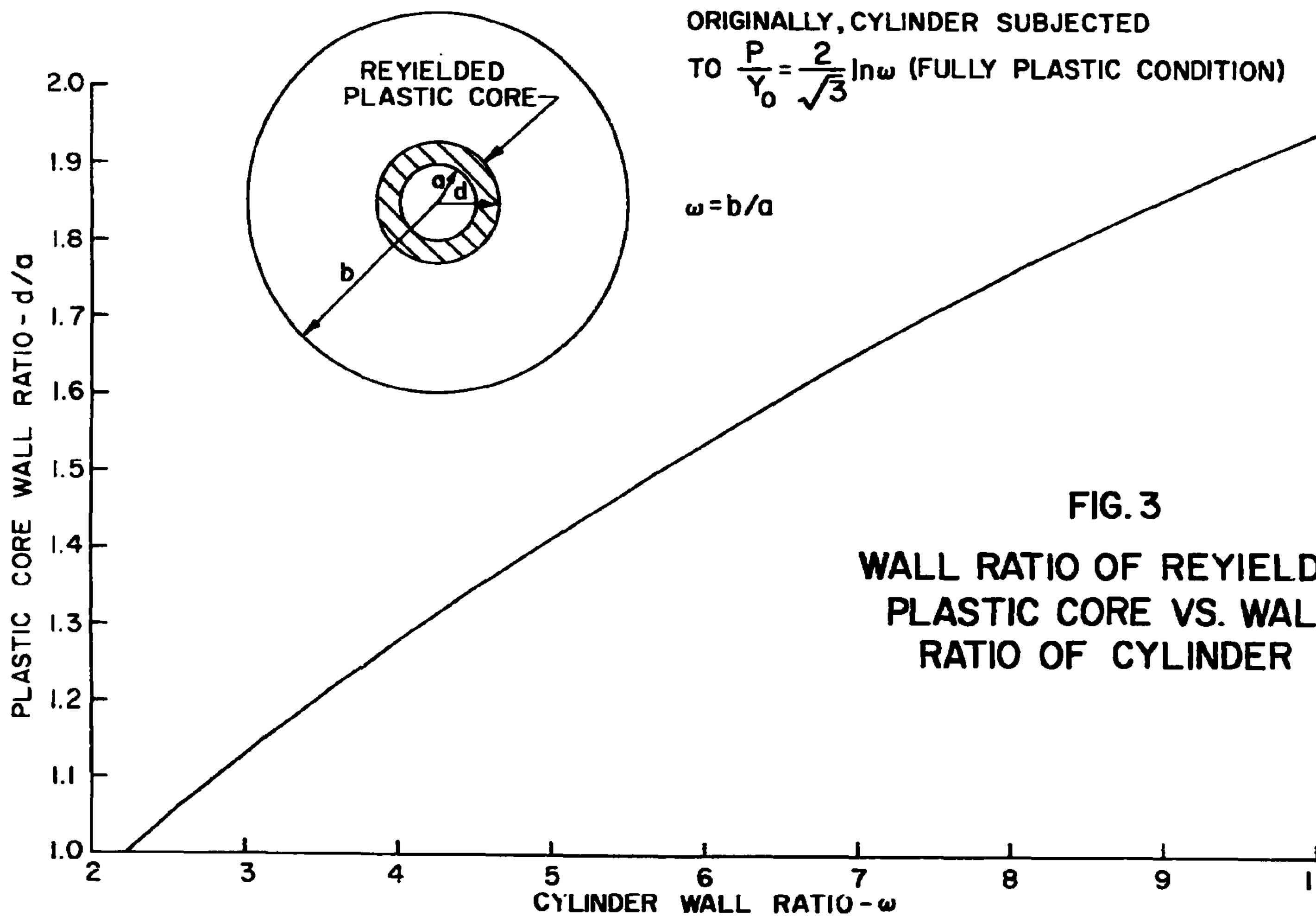


FIG. 3

WALL RATIO OF REYIELDED PLASTIC CORE VS. WALL RATIO OF CYLINDER

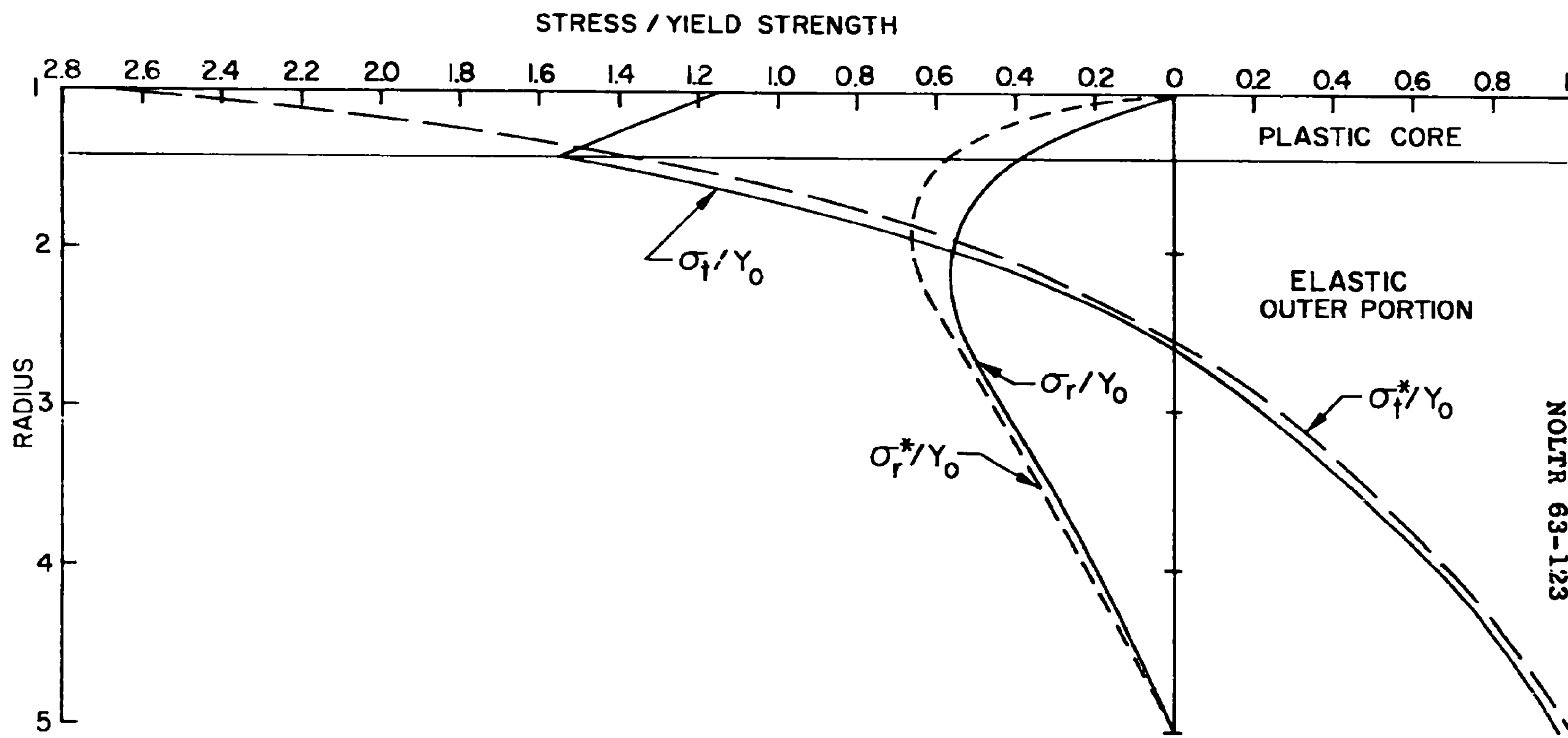


FIG. 4

STRESS DISTRIBUTION IN REYIELDED CYLINDER OF WALL RATIO 5

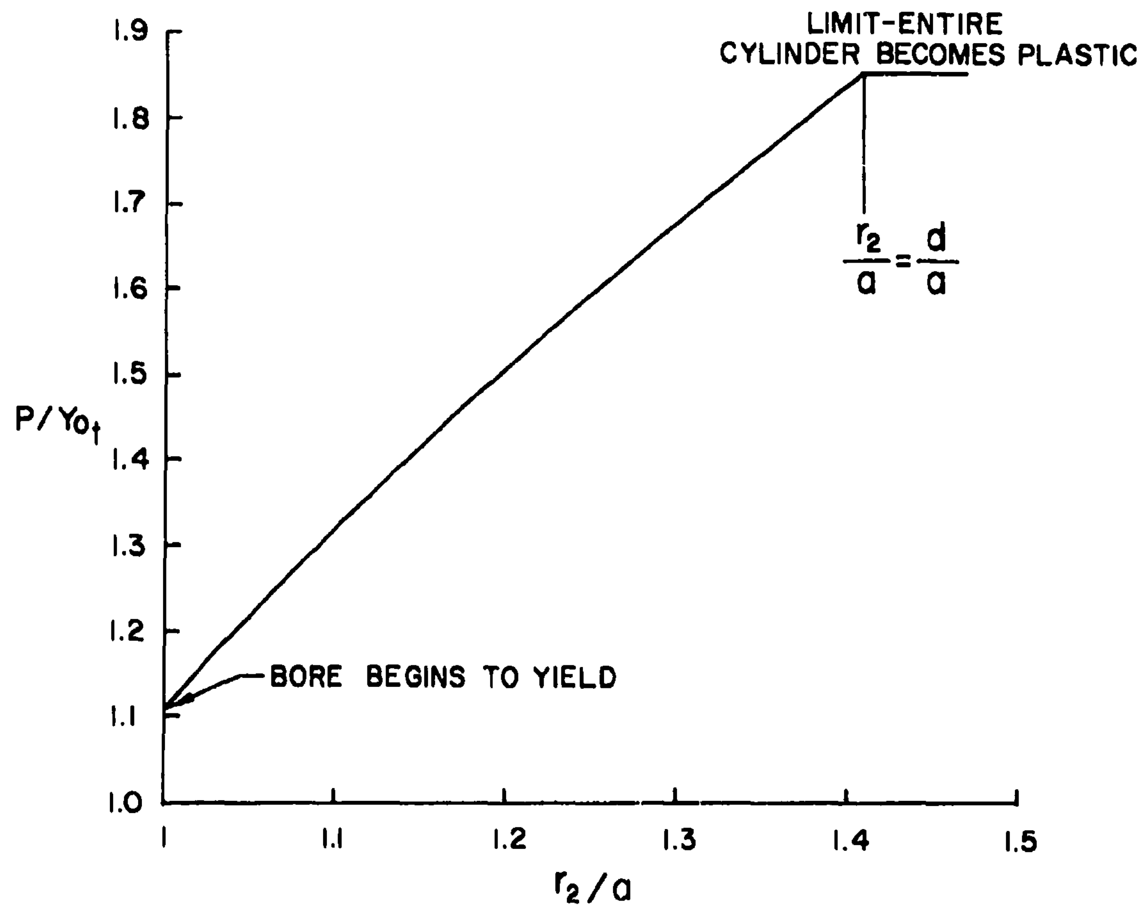


FIG. 5

PRESSURE REQUIRED TO CAUSE PLASTIC FLOW IN A REYIELDED CYLINDER OF WALL RATIO 5.

APPENDIX A

ALTERNATE DERIVATION OF THE REYIELDING EQUATIONS

When the pressure is released in a fully plastic large wall ratio tube, the cylinder will consist of a plastic reyielded center core and an outer elastic jacket. Thus, the core can be considered as a tube under external pressure, p, which has caused the core to be fully plastic. The outer jacket can be considered as an elastic tube with an internal pressure, p, which produces a final stress which is the sum of the residual and Lamé stresses.

For a tube subjected to external pressure the following equations apply

$$\sigma_t = -\frac{p}{\omega^2-1}\left(\frac{b^2}{r^2}+\omega^2\right) \tag{A-1}$$

$$\sigma_r = -\frac{p}{\omega^2-1}\left(\omega^2-\frac{b^2}{r^2}\right) \tag{A-2}$$

It is assumed that

$$\sigma_z = \frac{1}{2}\left(\sigma_t+\sigma_r\right) \tag{A-3}$$

The yield criterion is

$$\sigma_t - \sigma_r = \pm\frac{2Y_0}{\sqrt{3}} \tag{A-4}$$

Consider the external pressure, q, to increase on the cylinder until the bore begins to yield in compression. The boundary conditions for this inner core are at r = a

$$\sigma_t = \frac{-2q\,\omega^2}{\omega^2-1} \tag{A-5}$$

and

$$\sigma_r = 0 \qquad \text{(A-6)}$$

From (A-4) therefore

$$\frac{-2p\omega^2}{\omega^2 - 1} = \frac{2Y_{oc}}{\sqrt{3}}$$

$$p = -\frac{(\omega^2 - 1)}{\sqrt{3}\,\omega^2} Y_{oc} \qquad \text{(A-6)}$$

As the external pressure is increased the plastic zone spreads. In the plastic zone

$$\sigma_t - \sigma_r = \frac{2Y_{oc}}{\sqrt{3}}$$

and

$$r\frac{d\sigma_r}{dr} = \frac{2Y_{oc}}{\sqrt{3}}$$

Thus,

$$\sigma_r = C_1 + \frac{2Y_{oc}}{\sqrt{3}} \ln r \qquad \text{(A-7)}$$

$$\sigma_t = C_1 + \frac{2Y_{oc}}{\sqrt{3}}(\ln r + 1) \qquad \text{(A-8)}$$

For a fully plastic tube, at r = b, $\sigma_r = -p$ and at r = a, $\sigma_r = 0$

$$-p = C_1 + \frac{2Y_{oc}}{\sqrt{3}} \ln b$$

$$C_1 = -p - \frac{2Y_{oc}}{\sqrt{3}} \ln b \qquad \text{(A-9)}$$

Thus,

$$\sigma_r = -p - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{b}{r} \qquad \text{(A-10)}$$

$$\sigma_t = -p - \frac{2Y_{oc}}{\sqrt{3}} \left(\ln \frac{b}{r} + 1 \right) \qquad \text{(A-11)}$$

To cause plastic flow throughout the tube

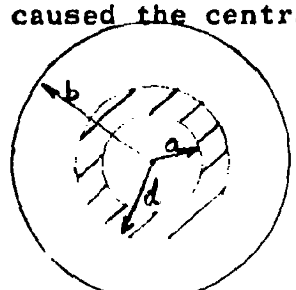
$$p = - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{b}{a} \qquad \text{(A-12)}$$

Therefore, in the fully plastic tube under external pressure

$$\sigma_r = \frac{2Y_{oc}}{\sqrt{3}} \ln \omega - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{b}{r} \qquad \text{(A-13)}$$

$$\sigma_t = \frac{2Y_{oc}}{\sqrt{3}} \ln \omega - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{b}{r} + \frac{2Y_{oc}}{\sqrt{3}} \qquad \text{(A-14)}$$

Consider now the case where reverse yielding occurs. On the outside of the plastic core there is a pressure, q, which has caused the central core to be plastic



To cause plastic flow throughout the core, from (A-12)

$$q = - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{d}{a} \qquad \text{(A-15)}$$

In the core the stress distribution is from (A-13) and (A-14)

$$\sigma_r = -q - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{d}{r}$$

$$= \frac{2Y_{oc}}{\sqrt{3}} \left(\ln \frac{d}{a} - \ln \frac{d}{r} \right)$$

$$\sigma_r = \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{r}{a}$$ (A-16)
See equation (19)

$$\sigma_t = -q - \frac{2Y_{oc}}{\sqrt{3}} \ln \frac{d}{r} + \frac{2Y_{oc}}{\sqrt{3}}$$

$$= \frac{2Y_{oc}}{\sqrt{3}} \left(\ln \frac{r}{a} + 1 \right)$$ (A-17)
See equation (20)

In the elastic part of the tube the residual elastic stresses will be the Lamé stresses plus the residual stresses created by the autofrettage process. Thus,

$$\sigma_t' = C_2 + \frac{C_3}{r^2} + \sigma_t^*$$ (A-18)

$$\sigma_r' = C_2 - \frac{C_3}{r^2} + \sigma_r^*$$ (A-19)

At r = d, the yield criterion holds so that

$$\sigma_{t_d}^* - \sigma_{r_d}^* + \frac{2C_3}{d^2} = \frac{2Y_{oc}}{\sqrt{3}}$$ (A-20)

From reference (3) or equations (3) and (4) with $p = \frac{2Y_o}{\sqrt{3}} \ln \omega$

$$\sigma_{t_d}^* - \sigma_{r_d}^* = -\frac{4Y_{ot}}{\sqrt{3}} \frac{b^2}{d^2} \frac{\ln \omega}{\omega^2 - 1} + \frac{2Y_{ot}}{\sqrt{3}}$$

Hence, from (A-20) with $Yo_c = -Y_{ot}$

$$C_3 = \frac{2Y_{oc}}{\sqrt{3}} d^2\left(1 - \frac{b^2 \ln\omega}{d^2(\omega^2-1)}\right) \tag{A-21}$$

Substituting c_3 in (A-19), and noting that at $r = b, \sigma_r^* = \sigma_r = 0$, gives

$$\sigma_r = 0 = C_2 - \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{b^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right)$$

so that

$$C_2 = \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{b^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right) \tag{A-22}$$

σ_r' and σ_t' can be written as

$$\sigma_r' = \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{b^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right) - \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{r^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right) + \sigma^*_{r\,r>d} \tag{A-23}$$

$$\sigma_t' = \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{b^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right) + \frac{2Y_{oc}}{\sqrt{3}} \frac{d^2}{r^2}\left(1 - \frac{b^2}{d^2} \frac{\ln\omega}{\omega^2-1}\right) + \sigma^*_{t\,r>d} \tag{A-24}$$

For the fully plastic core, from (A-16)

$$\sigma_r = \frac{2Y_{oc}}{\sqrt{3}} \ln\frac{d}{a}$$

This must equal σ_r' at $r = d$, so that

$$\frac{2Y_{oc}}{\sqrt{3}}\frac{d^2}{b^2}\left(1-\frac{b^2}{d^2}\frac{\ln\omega}{\omega^2-1}\right)\left(\frac{d^2}{b^2}-1\right)+\frac{2Y_{oc}}{\sqrt{3}}\left(\ln\frac{b}{d}+\frac{\ln\omega}{\omega^2-1}\left(1-\frac{b^2}{d^2}\right)\right)$$

$$=\frac{2Y_{oc}}{\sqrt{3}}\ln\frac{d}{a}$$

Solving this gives

$$\ln\frac{d}{a}=\frac{d^2}{b^2}-1+\ln\frac{b}{d} \qquad \text{(A-25)}$$

which is the same as (27).

Using (A-25) and the values of σ_t^* and σ_r^*, the final residual stresses in the elastic part of the reyielded cylinder can be written

$$\sigma_r'=-\frac{2Y_0}{\sqrt{3}}\left(\ln\frac{b}{r}-\left(\frac{d}{r}\right)^2+\left(\frac{d}{b}\right)^2\right) \qquad \text{(A-26)}$$

$$\sigma_t'=-\frac{2Y_0}{\sqrt{3}}\left(\ln\frac{b}{r}-1+\left(\frac{d}{r}\right)^2+\left(\frac{d}{b}\right)^2\right) \qquad \text{(A-27)}$$

which are the same as (28) and (29).

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The equations are developed for the case of a reverse yielded thick-walled cylinder. It is assumed that a cylinder is subjected to an internal pressure which causes plastic flow throughout the wall; the size of the cylinder is such that the residual stresses developed during pressure release cause the cylinder to reyield in compression. The stress equations for the subsequent reapplication of pressure to the reyielded cylinder are also developed.

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